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A separable form for the multichannel T matrix

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Abstract. The T matrix for a multichannel system with an arbitrary number of two-body channels is obtained in the form

$$T_{\beta\alpha}(p_\beta, p_\alpha, s) = G^\mu_\beta(p_\beta)\mathcal{T}^{\mu\nu}(s)G^\nu_\alpha(p_\alpha).$$

The result satisfies multichannel unitarity and allows a parametrization of the system in terms of its bound states and resonances. The relevance of this result to calculations on the few-body problem is discussed.

1. Introduction

In recent years considerable progress has been made in the formulation of the few-body problem and a most useful current review has been given by Sandhas (1976). As is well known from the work of Faddeev (1961a, b) the three-body problem can be formulated in terms of the solutions to the two-body subsystems and the non-trivial generalization of this formulates the N-body system equations in terms of the solutions to the n-body subsystems ($n < N$). The exact solution of such equations with N as small as four is a formidable task and it is clear that in the initial stages some approximation scheme is desirable. One scheme, suggested by Grassberger and Sandhas (1967), is based upon the pole approximation to the n-body subsystems. This has the advantage of a separable form for the n-body operators reducing the N-body equations to effective two-body Lippmann–Schwinger equations (a considerable simplification from a computational point of view).

The present paper shows how such a separable form for the subsystem can be obtained from a multichannel system with separable potentials. The final result is independent of the potential form factors and is expressed in terms of the form factors associated with the bound states and resonances of the subsystem. This form satisfies off-shell multichannel unitarity exactly and the separable structure ensures that higher-order N-body equations can be reduced to quasi-two-body equations.

2. The multichannel system

The multichannel system will consist of a series of two-body channels labelled by α with threshold energy E_α and momentum p_α. With an appropriate choice of units (Lovelace 1964a, b) the energy can be written as $s_\alpha = E_\alpha + p_\alpha^2$.

For simplicity we will assume that all the particles in the channels are spinless and consider scattering in a state of total angular momentum J. This implies that the relative angular momentum in each channel is also J. This simplifies the presentation of the essential features but the result is (for once) readily generalized to include particles with spin.

The model multichannel Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{1}$$

is assumed to have a separable potential with

$$\langle \beta p_\beta JM | \hat{H}_0 | \alpha p_\alpha JM \rangle = \delta_{\alpha\beta} \frac{\delta(p_\alpha - p_\beta)}{4\pi p_\alpha^2} (p_\alpha^2 + E_\alpha)$$

and

$$\langle \beta p_\beta JM | \hat{V} | \alpha p_\alpha JM \rangle = \sum_{ij} \lambda^{ij}_{\beta\alpha} g_{\beta i}(p_\beta) g_{\alpha j}(p_\alpha).$$

The index i labels the separable terms associated with channel β and strictly speaking it should be $i_\beta = 1, \ldots, N_\beta$ and $j_\alpha = 1, \ldots, N_\alpha$. For convenience the channel index is suppressed and in the following a summation is *implied* if a Latin index is repeated whereas a summation over the channel index α, β, γ only occurs if shown *explicitly*.

The T matrix is

$$T_{\beta\alpha}(p_\beta, p_\alpha, s) = g_{\beta i}(p_\beta) \tau^{ij}_{\beta\alpha} g_{\alpha j}(p_\alpha) \tag{2}$$

with

$$(\tau^{-1})^{ij}_{\beta\alpha} = (\lambda^{-1})^{ij}_{\beta\alpha} + \delta_{\alpha\beta} 4\pi \int \frac{g_{\beta i}(p_\beta) g_{\beta j}(p_\beta) p_\beta^2}{p_\beta^2 + E_\beta - s} \, dp_\beta$$

and this can be transformed using the procedure outlined previously (Preist 1977) to exhibit the bound state and resonance contributions. This employs Lovelace's continuation Lovelace (1964a, b) but the extension to the multichannel problem is further complicated by the cut structure in the complex energy-plane.

The physical region, $s = k^2 + i\epsilon$ with $k^2 \geq \min(E_\alpha)$, has the series of overlapping cuts shown in figure 1. The cuts start at the channel thresholds $E_1, E_2, \ldots$ and the physical value is obtained by approaching the overlapping cuts from above as shown in figure 1. The main complication arises because the continuation from the interval between two successive thresholds occurs on a different Reimann sheet for each such pair of thresholds. This is shown in figure 2 for the continuation from the interval $E_n < k^2 <$

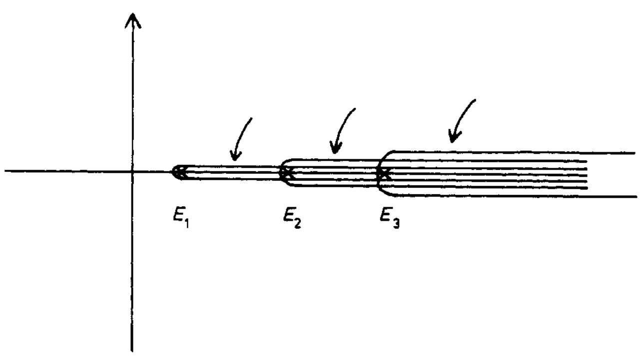

Figure 1. The complex energy-plane.

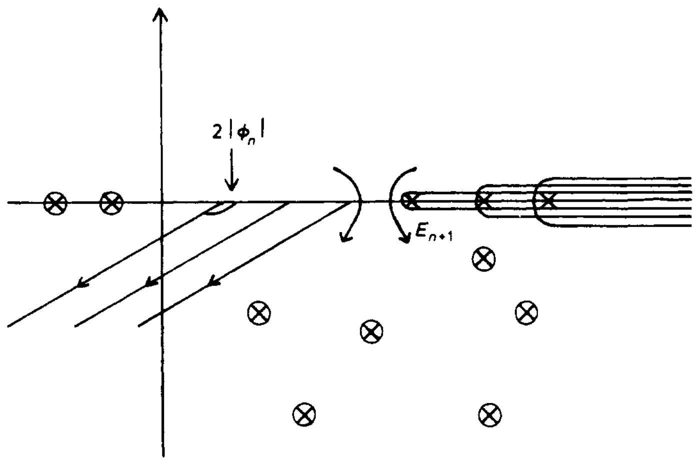

Figure 2. The *n*th Reimann sheet.

E_{n+1}. The cut associated with the threshold E_n and the cuts from all the lower thresholds are rotated through a (negative) angle $2\phi_n$ into the lower half-plane; $2|\phi_n|$ is sufficiently large to expose all the singularities in the lower half of this Reimann sheet (see figure 2).

Lovelace (1964a, b) has shown that these singularities are eigenvalues of the complex potential problem obtained from (1) by the replacement $p_\alpha \to p_\alpha\, e^{i\phi_\alpha}$ and $\lambda^{ij}_{\beta\alpha} \to \lambda^{ij}_{\beta\alpha}\, e^{i(\phi_\alpha+\phi_\beta)}$ with

$$\phi_\alpha = \begin{cases} \phi_n & \text{if } E_\alpha \leqslant E_n \\ 0 & \text{if } E_\alpha > E_n. \end{cases}$$

The corresponding T matrix is

$$T^n_{\beta\alpha}(p_\beta, p_\alpha, s) = G^\mu_{n\beta}(p_\beta)\tau^{\mu\nu}_n(s)G^\nu_{n\alpha}(p_\alpha) \tag{3}$$

with

$$(\tau_n^{-1}(s))^{\mu\nu} = \delta_{\mu\nu}(s - s_{n\mu}) + 4\pi(s - s_{n\mu})(s - s_{n\nu})\sum_\gamma \int \frac{\psi^\mu_{n\gamma}(p_\gamma)\psi^\nu_{n\gamma}(p_\gamma)p_\gamma^2\, dp_\gamma}{p_\gamma^2\, e^{2i\phi_\gamma} + E_\gamma - s}.$$

The $\psi^\mu_{n\gamma}(p_\gamma)$ are the components of the wavefunction associated with the real or complex eigenvalue $s_{n\mu}$ and the corresponding form factors are

$$G^\mu_{n\gamma}(p_\gamma) = (s_{n\mu} - E_\gamma - p_\gamma^2\, e^{2i\phi_\gamma})\psi^\mu_{n\gamma}(p_\gamma).$$

It should be noted that the pole associated with a resonance appears at a slightly different position $s_{n\mu}$ on each of the Reimann sheets. This point has been discussed in some detail by Dalitz and Rajasekaran (1963), Eden and Taylor (1963) and by Ross (1963); a resonance is associated with a zero in the inverse K matrix and this produces a family of poles on the various Reimann sheets.

The form factors may be continued to $\phi_\gamma = 0$ according to

$$G^\mu_{n\gamma}(p_\gamma) = e^{3i\phi_\gamma/2}G^\mu_\gamma(p_\gamma\, e^{i\phi_\gamma})$$

so that an alternative form for τ_n is

$$(\tau_n^{-1}(s))^{\mu\nu} = \delta_{\mu\nu}(s - s_{n\mu}) + 4\pi(s - s_{n\mu})(s - s_{n\nu}) \times \sum_\gamma \oint_{C_{n\gamma}} \frac{G_\gamma^\mu(z_\gamma)G_\gamma^\nu(z_\gamma)z_\gamma^2\,dz_\gamma}{(z_\gamma^2 + E_\gamma - s_{n\mu})(z_\gamma^2 + E_\gamma - s_{n\nu})(z_\gamma^2 + E_\gamma - s)} \tag{4}$$

where the contour $C_{n\gamma}$ runs from E_γ along $\arg z_\gamma = \phi_n$ for $\gamma \leq n$ and $\arg z_\gamma = 0$ for $\gamma > n$.

For $\phi_n < 0$ Lovelace's argument gives

$$T_{\beta\alpha}^n(p_\beta, p_\alpha, k^2 + i\epsilon) = \exp[\tfrac{3}{2}i(\phi_\alpha + \phi_\beta)]T_{\beta\alpha}(p_\beta\, e^{i\phi_\beta}, p_\alpha\, e^{i\phi_\alpha}, k^2 + i\epsilon)$$

provided $E_n < k^2 < E_{n+1}$.

Hence in this energy interval

$$T_{\beta\alpha}(p_\beta, p_\alpha, k^2 + i\epsilon) = G_\beta^\mu(p_\beta)\tau_n^{\mu\nu}(k^2 + i\epsilon)G_\alpha^\nu(p_\alpha) \tag{5}$$

where $(\tau_n^{-1}(k^2 + i\epsilon))^{\mu\nu}$ is given by (4) with $s = k^2 + i\epsilon$ and ϕ_n taking a negative value sufficiently large in magnitude to expose the poles on the lower half of the nth Reimann sheet. The orthogonality condition on this sheet takes the form

$$4\pi \sum_\gamma \oint_{C_{n\gamma}} \frac{G_\gamma^\mu(z_\gamma)G_\gamma^\nu(z_\gamma)z_\gamma^2\,dz_\gamma}{(z_\gamma^2 + E_\gamma - s_{n\mu})(z_\gamma^2 + E_\gamma - s_{n\nu})} = \delta_{\mu\nu}. \tag{6}$$

In a similar way

$$T_{\beta\alpha}(p_\beta, p_\alpha, k^2 - i\epsilon) = G_\beta^\mu(p_\beta)\tau_n^{\mu\nu}(k^2 - i\epsilon)G_\alpha^\nu(p_\alpha) \tag{7}$$

with $\tau_n^{\mu\nu}(k^2 - i\epsilon)$ obtained from $\tau_n^{\mu\nu}(k^2 + i\epsilon)$ by the replacements $k^2 + i\epsilon \to k^2 - i\epsilon$, $\phi_n \to -\phi_n$ and $s_{n\mu} \to s_{n\mu}^*$. Off-shell unitarity in the region $E_n < k^2 < E_{n+1}$ may then be proved using the method outlined by Preist (1977).

An alternative form for τ^{-1} is

$$(\tau_n^{-1}(s))^{\mu\nu} = B^{\mu\nu} + 4\pi \sum_\gamma \oint_{C_{n\gamma}} \frac{G_\gamma^\mu(z_\gamma)G_\gamma^\nu(z_\gamma)z_\gamma^2\,dz_\gamma}{z_\gamma^2 + E_\gamma - s} \tag{8}$$

and from this the continuity condition readily follows, i.e.

$$(\tau_n^{-1}(E_{n+1}))^{\mu\nu} = (\tau_{n+1}^{-1}(E_{n+1}))^{\mu\nu}.$$

This particular form also allows the pole positions on successive sheets to be determined.

The poles occur on successive sheets at s_n and s_{n+1}, where

$$(\tau_{n+1}^{-1}(s_{n+1,\mu}))^{\mu\nu} = (\tau_n^{-1}(s_{n\mu}))^{\mu\nu} = 0.$$

Using this condition (with $\nu = \mu$) together with the above expression for τ^{-1} provides an implicit equation for the shift in the pole position; if this is small then

$$s_{n\mu} - s_{n+1,\mu} \approx 4\pi^2 i(G_n^\mu(z_{n\mu}))^2 z_{n\mu}$$

which is a result similar in form to that obtained by Dalitz and Rajasekaran (1963).

3. Summary

The preceding sections show how to describe a multichannel system in terms of its singularities and form factors associated with these singularities. The description is not simple, reflecting the complexity of the multichannel system, but it does possess two important and desirable features.

First it is separable with the separable terms associated with singularities in the complex energy-plane. Some singularities are readily identified with bound states and resonances whereas others, more distant from the physical region provide separable background terms.

Secondly multichannel unitarity is satisfied exactly for an arbitrary number of singularities with the same total angular momentum. This is in contrast to the usual sum of pole terms which violates unitarity when the poles are close together.

The penalty for not solving the subsystem problem *ab initio* is the introduction of a large number of arbitrary form factors. This feature is inherent in any attempt to short circuit the complete calculation and is in line with the suggestion of Mitra and Sharma (1976) to introduce semi-phenomenological vertex functions for the effective potentials in the 'two-body' equations describing the N-body system. The form factors are constrained by the orthogonality condition and in many cases physical information is available about the subsystems themselves. Such information may suggest the location of the singularities and reasonable forms for the form factors.

In a more ambitious programme one might use the separable form obtained as a basis for a variational solution to the subsystem problem obtaining a result which though approximate satisfied unitarity exactly.

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